Unit : 3

Class code : Ljfc3vk

Algebra

Exponential Series

Formulae:

$$* \ e^x = 1 + \frac{x}{|\underline{1}} + \frac{x^2}{|\underline{2}} + \frac{x^3}{|\underline{3}} + \cdots \infty$$

$$* \ e^{-x} = 1 - \frac{x}{|\underline{1}} + \frac{x^2}{|\underline{2}} - \frac{x^3}{|\underline{3}} + \cdots \infty$$

$$* \ \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{|\underline{2}} + \frac{x^4}{|\underline{4}} + \cdots \infty$$

$$* \ \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{|\underline{3}} + \frac{x^5}{|\underline{5}} + \cdots \infty$$

$$* \ e = 1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \cdots \infty$$

$$* \ e^{-1} = 1 - \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} - \frac{1}{|\underline{3}} + \cdots \infty$$

$$* \ \frac{e + e^{-1}}{2} = 1 + \frac{1}{|\underline{2}} + \frac{1}{|\underline{4}} + \cdots \infty$$

$$* \ \frac{e - e^{-1}}{2} = \frac{1}{|\underline{1}} + \frac{1}{|\underline{3}} + \frac{1}{|\underline{5}} + \cdots \infty$$

Notes:

$$* \ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$* \ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$* \ 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 1

Find the coefficient of x^n in the expansion of e^{a+bx}

$$e^{a+bx} = e^a . e^{bx}$$

$$= e^a \left[1 + \frac{bx}{\underline{|1}} + \frac{(bx)^2}{\underline{|2}} + \dots + \frac{(bx)^n}{\underline{|n}} + \dots \infty \right]$$

Coefficient of x^n in this expansion $= \dfrac{e^a . b^n}{\underline{|n}}$

Example 2

Expand a^x in ascending powers of x, 'a' being positive.

$$a^x = e^{x \log a}$$

$$a^x = 1 + \frac{x \log a}{\underline{|1}} + \frac{x^2 (\log a)^2}{\underline{|2}} + \dots + \frac{x^n (\log a)^n}{\underline{|n}} + \dots \infty$$

Example 3

Prove that $\dfrac{e-1}{e+1} = \dfrac{\frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|6}} + \dots \infty}{\frac{1}{\underline{|1}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \dots \infty}$

$$\text{RHS} = \frac{\frac{e+e^{-1}}{2} - 1}{\frac{e-e^{-1}}{2}}$$

$$= \frac{\frac{e-e^{-1}-2}{\cancel{2}}}{\frac{e-e^{-1}}{\cancel{2}}}$$

$$= \frac{e-e^{-1}-2}{e-e^{-1}}$$

$$RHS = \frac{e - \frac{1}{e} - 2}{e - \frac{1}{e}}$$

$$= \frac{e^2 - 1 - 2e}{e^2 - 1}$$

$$= \frac{(e-1)^{\cancel{2}}}{(e+1)\cancel{(e-1)}}$$

$$RHS = \frac{e-1}{e+1}$$

Example 4

Show that

$$2\left[1 + \frac{(\log_e n)^2}{\lfloor 2} + \frac{(\log_e n)^4}{\lfloor 4} + \cdots \infty\right] = n + \frac{1}{n}$$

$$LHS = 2\left[1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots \infty\right] \quad \text{where } x = \log_e n$$

$$= \cancel{2}\frac{(e^x + e^{-x})}{\cancel{2}}$$

$$= e^x + e^{-x}$$

$$= e^{\log_e n} + e^{-\log_e n}$$

$$= e^{\log_e n} + \frac{1}{e^{\log_e n}}$$

$$= n + \frac{1}{n}$$

$$= RHS$$

Example 5

Sum to infinity the series

$$1 + \frac{1+2}{\lfloor 2} + \frac{1+2+2^2}{\lfloor 3} + \dots \infty$$

The nth term of the series is

$$T_n = \frac{1+2+2^2+\dots+2^{n-1}}{\lfloor n}$$

$$= \frac{2^n - 1}{\lfloor n}$$

$$= \frac{2^n}{\lfloor n} - \frac{1}{\lfloor n}$$

Putting $n = 1, 2, 3, \dots$ we get,

$$T_1 = \frac{2}{\lfloor 1} - \frac{1}{\lfloor 1}$$

$$T_2 = \frac{2^2}{\lfloor 2} - \frac{1}{\lfloor 2}$$

$$T_3 = \frac{2^3}{\lfloor 3} - \frac{1}{\lfloor 3}$$

....

....

Adding

$$S_\infty = \left(\frac{2}{\lfloor 1} + \frac{2^2}{\lfloor 2} + \frac{2^3}{\lfloor 3} + \dots \infty\right) - \left(\frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots \infty\right)$$

$$= (e^2 - 1) - (e - 1)$$

$$= e^2 - \cancel{1} - e + \cancel{1}$$

$$S_\infty = e(e-1)$$

Example 6

Sum to infinity the series

$$\frac{1^2}{\lfloor 1} + \frac{1^2+2^2}{\lfloor 2} + \frac{1^2+2^2+3^2}{\lfloor 3} + \dots \infty$$

The nth term of the series is given by

$$T_n = \frac{1^2+2^2+3^2+\dots+n^2}{\lfloor n}$$

$$= \frac{n(n+1)(2n+1)}{6\lfloor n}$$

$$= \frac{\cancel{n}(n+1)(2n+1)}{6(\cancel{n})\lfloor n-1}$$

$$= \frac{(n+1)(2n+1)}{6\lfloor n-1}$$

Let $(n+1)(2n+1) = A + B(n-1) + C(n-1)(n-2)$

Put $n=1$;

$(2)(3) = A + B(0) + C(0)$

$\Rightarrow \boxed{A = 6}$

Put $n=2$;

$(3)(5) = A + B + C(0)$

$\Rightarrow A + B = 15$

$B = 15 - 6$

$\boxed{B = 9}$

Equating the coefficient of n^2,

$\boxed{C = 2}$

$$\therefore \; T_n = \frac{6 + 9(n-1) + 2(n-1)(n-2)}{6\lfloor n-1}$$

$$= \frac{\cancel{6}}{\cancel{6}\lfloor n-1} + \frac{{}^{3}\cancel{9}(n-1)}{{}_{2}\cancel{6}\lfloor n-1} + \frac{\cancel{2}\cancel{(n-1)}\cancel{(n-2)}}{\cancel{6}_{3}\cancel{(n-1)}\cancel{(n-2)}\lfloor n-3}$$

$$= \frac{1}{\lfloor n-1} + \frac{3\cancel{(n-1)}}{2\cancel{(n-1)}\lfloor n-2} + \frac{1}{3\lfloor n-3}$$

$$T_n = \frac{1}{\lfloor n-1} + \frac{3}{2\lfloor n-2} + \frac{1}{3\lfloor n-3}$$

Putting $n = 1, 2, 3, \ldots$ we get,

$$T_1 = 1$$

$$T_2 = \frac{1}{\lfloor 1} + \frac{3}{2}$$

$$T_3 = \frac{1}{\lfloor 2} + \frac{3}{2\lfloor 1} + \frac{1}{3}$$

$$T_4 = \frac{1}{\lfloor 3} + \frac{3}{2\lfloor 2} + \frac{1}{3\lfloor 1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Adding $S_\infty = \left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \ldots \infty\right)$

$$+ \frac{3}{2}\left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \ldots \infty\right) + \frac{1}{3}\left(1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots \infty\right)$$

$$S_\infty = e + \frac{3}{2}e + \frac{1}{3}e$$

$$= \frac{6e + 9e + 2e}{6}$$

$$S_\infty = \frac{17e}{6}$$

Example 7

Find the sum to infinity of the series

$$1 + \frac{2^3}{\lfloor 2} + \frac{3^3}{\lfloor 3} + \frac{4^3}{\lfloor 4} + \dots \infty$$

The nth term of the series in

$$T_n = \frac{n^3}{\lfloor n}$$

$$= \frac{n^3}{n\lfloor n-1}$$

$$T_n = \frac{n^2}{\lfloor n-1}$$

Let $n^2 = A + B(n-1) + C(n-1)(n-2)$

Put $n = 1$;

$$1^2 = A + B(0) + C(0)$$

$$\Rightarrow \boxed{A = 1}$$

Put $n = 2$;

$$2^2 = A + B(1) + C(0)$$

$$\Rightarrow A + B = 4$$

$$B = 4 - 1$$

$$\boxed{B = 3}$$

Equating the coefficient of n^2, $\boxed{C = 1}$

$$\therefore T_n = \frac{1 + 3(n-1) + 1(n-1)(n-2)}{\lfloor n-1}$$

$$T_n = \frac{1}{\lfloor n-1} + \frac{3(n-1)}{(n-1)\lfloor n-2} + \frac{(n-1)(n-2)}{(n-1)(n-2)\lfloor n-3}$$

$$T_n = \frac{1}{\lfloor n-1} + \frac{3}{\lfloor n-2} + \frac{1}{\lfloor n-3}$$

$$T_n = \frac{1}{\lfloor n-1} + \frac{3}{\lfloor n-2} + \frac{1}{\lfloor n-3}$$

Putting $n = 1, 2, 3, \ldots$ we get,

$$T_1 = 1$$

$$T_2 = \frac{1}{\lfloor 1} + 3$$

$$T_3 = \frac{1}{\lfloor 2} + \frac{3}{\lfloor 1} + 1$$

$$T_4 = \frac{1}{\lfloor 3} + \frac{3}{\lfloor 2} + \frac{1}{\lfloor 1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Adding $$S_\infty = \left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots \infty\right) + 3\left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots \infty\right) + \left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots \infty\right)$$

$$= e + 3e + e$$

$$S_\infty = 5e$$

Example 8

Sum to infinity the series

$$1 + \frac{2^4}{\lfloor 2} + \frac{3^4}{\lfloor 3} + \frac{4^4}{\lfloor 4} + \cdots \infty$$

The nth term of the series is

$$T_n = \frac{n^4}{\lfloor n}$$

$$= \frac{n^{\cancel{4}3}}{\cancel{n}\lfloor n-1}$$

$$T_n = \frac{n^3}{\lfloor n-1}$$

Let $n^3 = A + B(n-1) + C(n-1)(n-2) + D(n-1)(n-2)(n-3)$

Put $n = 1$;

$1^3 = A$

$\Rightarrow \boxed{A = 1}$

Put $n = 2$;

$2^3 = A + B(1)$

$\Rightarrow A + B = 8$

$\boxed{B = 7}$

Put $n = 3$;

$3^3 = A + B(2) + C(2)(1)$

$\Rightarrow A + 2B + 2C = 27$

$1 + 14 + 2C = 27$

$2C = 27 - 15$

$C = \frac{12}{2}$

$\boxed{C = 6}$

$\therefore$ Equating the coefficient of n^3, $\boxed{D = 1}$

$$T_n = \frac{1 + 7(n-1) + 6(n-1)(n-2) + 1(n-1)(n-2)(n-3)}{\underline{|n-1}}$$

$$T_n = \frac{1}{\underline{|n-1}} + \frac{7\cancel{(n-1)}}{\cancel{(n-1)}\underline{|n-2}} + \frac{6\cancel{(n-1)}\cancel{(n-2)}}{\cancel{(n-1)}\cancel{(n-2)}\underline{|n-3}} + \frac{\cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}}{\cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}\underline{|n-4}}$$

$$T_n = \frac{1}{\underline{|n-1}} + \frac{7}{\underline{|n-2}} + \frac{6}{\underline{|n-3}} + \frac{1}{\underline{|n-4}}$$

Putting $n = 1, 2, 3, \cdots$ and adding

$$S_\infty = \sum_{n=1}^{\infty} \frac{1}{\lfloor n-1} + 7\sum_{n=2}^{\infty} \frac{1}{\lfloor n-2} + 6\sum_{n=3}^{\infty} \frac{1}{\lfloor n-3}$$

$$+ \sum_{n=4}^{\infty} \frac{1}{\lfloor n-4}$$

$$= e + 7e + 6e + e$$

$$S_\infty = 15e$$

Example 9

Sum to infinity the series

$$\frac{1^2 \cdot 2^2}{\lfloor 1} + \frac{2^2 \cdot 3^2}{\lfloor 2} + \frac{3^2 \cdot 4^2}{\lfloor 3} + \dots \infty$$

The nth term of the series is

$$T_n = \frac{n^2 (n+1)^2}{\lfloor n}$$

$$T_n = \frac{n(n+1)^2}{\lfloor n-1}$$

Let $n(n+1)^2 = A + B(n-1) + C(n-1)(n-2)$
$+ D(n-1)(n-2)(n-3)$

Put $n = 1$;

$$1(1+1)^2 = A$$

$$\Rightarrow \boxed{A = 4}$$

Put $n = 2$;

$$2(3)^2 = A + B(1)$$

$$\Rightarrow A + B = 2(9)$$

$$4 + B = 18$$

$$\boxed{B = 14}$$

Put $n = 3$;

$$3(4)^2 = A + B(2) + C(2)(1)$$

$$3(16) = A + 2B + 2C$$

$$\Rightarrow A + 2B + 2C = 48$$

$$4 + 28 + 2C = 48$$

$$2C = 48 - 32$$

$$C = \frac{16}{2}$$

$$\boxed{C = 8}$$

Equating the coefficient of n^3, $\boxed{D = 1}$

$$T_n = \frac{4 + 14(n-1) + 8(n-1)(n-2) + (n-1)(n-2)(n-3)}{|\underline{n-1}}$$

$$= \frac{4}{|\underline{n-1}} + \frac{14}{|\underline{n-2}} + \frac{8}{|\underline{n-3}} + \frac{1}{|\underline{n-4}}$$

Putting $n = 1, 2, 3, \cdots \infty$ we get,

$$S_\infty = 4\sum_{n=1}^{\infty} \frac{1}{|\underline{n-1}} + 14\sum_{n=2}^{\infty} \frac{1}{|\underline{n-2}} + 8\sum_{n=3}^{\infty} \frac{1}{|\underline{n-3}} + \sum_{n=4}^{\infty} \frac{1}{|\underline{n-4}}$$

$$= 4\left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \cdots \infty\right) + 14\left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \cdots \infty\right)$$

$$+ 8\left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \cdots \infty\right) + \left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \cdots \infty\right)$$

$$S_\infty = 4e + 14e + 8e + e$$

$$= 18e + 9e$$

$$S_\infty = 27e$$

Example 10

Sum to infinity the series,

$$5 + \frac{2.6}{\lfloor 1} + \frac{3.7}{\lfloor 2} + \frac{4.8}{\lfloor 3} + \dots \infty$$

The given series is

$$\frac{1.5}{\lfloor 0} + \frac{2.6}{\lfloor 1} + \frac{3.7}{\lfloor 2} + \dots \infty$$

$$T_n = \frac{n(n+4)}{\lfloor n-1}$$

Let $n(n+4) = A + B(n-1) + C(n-1)(n-2)$

Put $n = 1$;

$1(1+4) = A$

$\Rightarrow \boxed{A = 5}$

Put $n = 2$;

$2(2+4) = A + B(1)$

$12 = 5 + B$

$\Rightarrow \boxed{B = 7}$

Equating the coefficient of n^2, $\boxed{C = 1}$

$$\therefore T_n = \frac{5}{\lfloor n-1} + \frac{7}{\lfloor n-2} + \frac{1}{\lfloor n-3}$$

$$\therefore S_\infty = 5 \sum_{n=1}^{\infty} \frac{1}{\lfloor n-1} + 7 \sum_{n=2}^{\infty} \frac{1}{\lfloor n-2} + \sum_{n=3}^{\infty} \frac{1}{\lfloor n-3}$$

$$= 5e + 7e + e$$

$$S_\infty = 13e.$$

Example 11

Sum to infinity the series

$$2^2 + \frac{3^2}{\underline{|1}}x + \frac{4^2}{\underline{|2}}x^2 + \frac{5^2}{\underline{|3}}x^3 + \dots \infty$$

$$T_n = \frac{(n+1)^2 x^{n-1}}{n-1}$$

Let $(n+1)^2 = A + B(n-1) + C(n-1)(n-2)$

Put $n = 1$;

$(1+1)^2 = A$

$\Rightarrow \boxed{A = 4}$

Put $n = 2$;

$(2+1)^2 = A + B(1)$

$\Rightarrow A + B = 9$

$\boxed{B = 5}$

Equating the coefficient of n^2, $\boxed{C = 1}$

$$\therefore T_n = \left(\frac{4}{\underline{|n-1}} + \frac{5}{\underline{|n-2}} + \frac{1}{\underline{|n-3}} \right) x^{n-1}$$

$$S_\infty = \sum_{n=1}^{\infty} T_n$$

$$= 4\sum_{n=1}^{\infty} \frac{x^{n-1}}{\underline{|n-1}} + 5x \sum_{n=2}^{\infty} \frac{x^{n-2}}{\underline{|n-2}} + x^2 \sum_{n=3}^{\infty} \frac{x^{n-3}}{\underline{|n-3}}$$

$$= 4\left(1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \dots \infty\right) + 5x\left(1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \dots \infty\right)$$

$$+ x^2\left(1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \dots \infty\right)$$

$$= 4e^x + 5xe^x + x^2e^x$$

$$S_\infty = (4 + 5x + x^2)\, e^x.$$

Example 12

Sum to infinity the series

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n+2} \frac{x^n}{\lfloor\underline{n}}$$

$$\frac{n^2+1}{(n+2)\lfloor\underline{n}} = \frac{(n^2+1)\,(n+1)}{(n+2)(n+1)\lfloor\underline{n}}$$

$$= \frac{(n^2+1)\,(n+1)}{\lfloor\underline{n+2}}$$

Let $(n^2+1)(n+1) = A+B(n+2)+C(n+2)(n+1)$
$+ D(n+2)(n+1)\,n$

Put $n=-2$;

$(4+1)(-2+1) = A$

$\Rightarrow A = (5)(-1)$

$\boxed{A=-5}$

Put $n=-1$;

$(2)(0) = A+B(1)$

$\Rightarrow B = -A$

$\boxed{B=5}$

Put $n=0$;

$1 = A+B(2)+C(2)(1)$

$\Rightarrow A+2B+2C = \cancel{0}\ 1$

$-5+10+2C = \cancel{0}\ 1$

$5+2C = 1$

$2C = -4$

$\boxed{C=-2}$

Equating the coefficient of n^3, $\boxed{D=1}$

$$\therefore T_n = \left(\frac{-5}{\lfloor n+2} + \frac{5}{\lfloor n+1} - \frac{2}{\lfloor n} + \frac{1}{\lfloor n-1}\right)x^n$$

$$= \frac{-5 \cdot x^{n+2}}{x^2 \lfloor n+2} + \frac{5}{x}\frac{x^{n+1}}{\lfloor n+1} - 2 \cdot \frac{x^n}{\lfloor n} + \frac{x \cdot x^{n-1}}{\lfloor n-1}$$

$$\therefore S_\infty = \sum T_n$$

$$= -\frac{5}{x^2}\sum_{n=1}^{\infty}\frac{x^{n+2}}{\lfloor n+2} + \frac{5}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{\lfloor n+1}$$
$$-2\sum_{n=1}^{\infty}\frac{x^n}{\lfloor n} + x\sum_{n=1}^{\infty}\frac{x^{n-1}}{\lfloor n-1}$$

$$= \frac{-5}{x^2}\left(\frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \dots \infty\right) + \frac{5}{x}\left(\frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \dots \infty\right)$$
$$-2\left(\frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots \infty\right) + x\left(1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \dots \infty\right)$$

$$= \frac{-5}{x^2}\left(e^x - 1 - x - \frac{x^2}{\lfloor 2}\right) + \frac{5}{x}(e^x - 1 - x) - 2(e^x - 1) + xe^x$$

$$= \frac{-5e^x}{x^2} + \frac{5}{x^2} + \frac{5x}{x^2} + \frac{5x^2}{\lfloor 2 x^2} + \frac{5e^x}{x} - \frac{5}{x} - \frac{5x}{x} - 2e^x$$
$$+2 + xe^x$$

$$= e^x\left(\frac{-5}{x^2} + \frac{5}{x} - 2 + x\right) + \frac{5}{x^2} + \frac{5}{x} + \frac{5}{2} - \frac{5}{x} - 5 + 2$$

$$= e^x\left(\frac{-5 + 5x - 2x^2 + x^3}{x^2}\right) + \frac{10 + \cancel{10x} + 5x^2 - \cancel{10x} - 10x^2 + 4x^2}{2x^2}$$

$$= e^x\left(\frac{x^3 - 2x^2 + 5x - 5}{x^2}\right) + \frac{10 - x^2}{2x^2}$$

Example 13

Sum to infinity the series

$$1 + \frac{2^3 x}{\lfloor\!\underline{1}} + \frac{3^3}{\lfloor\!\underline{2}} x^2 + \dots \infty$$

$$T_n = \frac{n^3}{\lfloor\!\underline{n-1}} x^{n-1}$$

Let

$$n^3 = A + B(n-1) + C(n-1)(n-2) + D(n-1)(n-2)(n-3)$$

Put $n = 1$;

$$\boxed{A = 1}$$

Put $n = 2$;

$$8 = A + B(1)$$

$$\Rightarrow B = 8 - 1$$

$$\boxed{B = 7}$$

Put $n = 3$;

$$27 = A + B(2) + C(2)(1)$$

$$\Rightarrow A = 27 - 2B - 2C$$

$$A = 27 - 14 - 2C$$

$$2C = 13 - 1$$

$$C = \frac{\cancel{12}^{\,6}}{\cancel{2}}$$

$$\boxed{C = 6}$$

Equating the coefficient of n^3, $\boxed{D = 1}$

$$\therefore T_n = \left(\frac{1}{\lfloor\!\underline{n-1}} + \frac{7}{\lfloor\!\underline{n-2}} + \frac{6}{\lfloor\!\underline{n-3}} + \frac{1}{\lfloor\!\underline{n-4}} \right) x^{n-1}$$

$$\therefore S_\infty = \sum_{n=1}^{\infty} \frac{x^{n-1}}{\lfloor n-1} + 7x \sum_{n=2}^{\infty} \frac{x^{n-2}}{\lfloor n-2} + 6x^2 \sum_{n=3}^{\infty} \frac{x^{n-3}}{\lfloor n-3}$$

$$+ x^3 \sum_{n=4}^{\infty} \frac{x^{n-4}}{\lfloor n-4}$$

$$S_\infty = e^x + 7xe^x + 6x^2 e^x + x^3 e^x$$

$$S_\infty = e^x (x^3 + 6x^2 + 7x + 1)$$

Example 14

Sum to infinity the series

$$\frac{3^2}{\lfloor 1} + \frac{5^2}{\lfloor 2} \log_e 2 + \frac{7^2}{\lfloor 3} (\log_e 2)^2 + \dots \infty$$

$$T_n = \frac{(2n+1)^2}{\lfloor n} (\log_e 2)^{n-1}$$

Let $(2n+1)^2 = A + Bn + C(n)(n-1)$

Put $n=0$;

$\boxed{A = 1}$

Put $n=1$;

$(2+1)^2 = A + B$

$\Rightarrow B = 9 - 1$

$\boxed{B = 8}$

Equating the coefficient of n^2; $\boxed{C = 4}$

$$\therefore T_n = \left(\frac{1}{\lfloor n} + \frac{8}{\lfloor n-1} + \frac{4}{\lfloor n-2} \right) x^{n-1} \text{ where } x = \log_e 2$$

$$= \frac{1}{x} \frac{x^n}{\lfloor n} + 8 \frac{x^{n-1}}{\lfloor n-1} + 4x \frac{x^{n-2}}{\lfloor n-2} ;$$

$$\therefore S_\infty = \sum_{n=1}^{\infty} T_n$$

$$= \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{\lfloor n} + 8\sum_{n=1}^{\infty}\frac{x^{n-1}}{\lfloor n-1} + 4x\sum_{n=2}^{\infty}\frac{x^{n-2}}{\lfloor n-2}$$

$$= \frac{1}{x}(e^x-1) + 8e^x + 4xe^x$$

$$= e^x\left(4x+8+\frac{1}{x}\right) - \frac{1}{x}$$

$$= e^{(\log_e 2)}\left[(4\log_e 2)+8+\frac{1}{(\log_e 2)}\right] - \frac{1}{\log_e 2}$$

$$= 2\left[4(\log_e 2)+8+\frac{1}{(\log_e 2)}\right] - \frac{1}{\log_e 2}$$

$$S_\infty = 8\log_e 2 + 16 + \frac{1}{\log_e 2}$$

Example 15

Sum to infinity the series

$$\sum_{n=0}^{\infty}\frac{5n+1}{\lfloor 2n+1}$$

Let $5n+1 = A+B(2n+1)$

Put $n = -\frac{1}{2}$;

$$-\frac{5}{2}+1 = A$$

$$\Rightarrow \boxed{A = \frac{-3}{2}}$$

Equating the coefficient of n, $\boxed{B = \frac{5}{2}}$

$$T_n = \frac{-3/2}{\lfloor 2n+1} + \frac{5/2\,(2n+1)}{\lfloor 2n+1}$$

$$T_n = -\frac{3}{2}\cdot\frac{1}{|\underline{2n+1}} + \frac{5}{2}\cdot\frac{(2n+1)}{(2n+1)|\underline{2n}}$$

$$T_n = -\frac{3}{2}\cdot\frac{1}{|\underline{2n+1}} + \frac{5}{2}\cdot\frac{1}{|\underline{2n}}$$

$$S_\infty = \sum T_n$$

$$= \frac{-3}{2}\sum_{n=0}^{\infty}\frac{1}{|\underline{2n+1}} + \frac{5}{2}\sum_{n=0}^{\infty}\frac{1}{|\underline{2n}}$$

$$\boxed{|\underline{n} = n|\underline{n-1}}$$

$$= \frac{-3}{2}\left(\frac{1}{|\underline{1}} + \frac{1}{|\underline{3}} + \frac{1}{|\underline{5}} + \cdots\infty\right) + \frac{5}{2}\left(1 + \frac{1}{|\underline{2}} + \frac{1}{|\underline{4}} + \cdots\infty\right)$$

$$= \frac{-3}{2}\left(\frac{e-e^{-1}}{2}\right) + \frac{5}{2}\left(\frac{e+e^{-1}}{2}\right)$$

$$= \frac{-3e}{4} + \frac{3e^{-1}}{4} + \frac{5e}{4} + \frac{5e^{-1}}{4}$$

$$= \frac{2e}{4} + \frac{8e^{-1}}{4}$$

$$S_\infty = \frac{e}{2} + \frac{2}{e}$$

Example 16

Sum to infinity the series

$$\frac{2.3}{|\underline{3}} + \frac{3.5}{|\underline{4}} + \frac{4.7}{|\underline{5}} + \cdots\infty$$

$$T_n = \frac{(n+1)(2n+1)}{|\underline{n+2}}$$

Let $(n+1)(2n+1) = A + B(n+2) + C(n+2)(n+1)$

Put $n=-2$;

$(-1)(-3) = A$

$\Rightarrow \boxed{A=3}$

Put $n=-1$;

$0 = 3 + B$

$\Rightarrow \boxed{B=-3}$

Equating the coefficient of n^2, $\therefore \boxed{C = 2}$

$$T_n = \frac{3}{\lfloor n+2} - \frac{3}{\lfloor n+1} + \frac{2}{\lfloor n}$$

$$\therefore S_\infty = \sum_{n=1}^{\infty} T_n$$

$$= 3\sum \frac{1}{\lfloor n+2} - 3\sum \frac{1}{\lfloor n+1} + 2\sum \frac{1}{\lfloor n}$$

$$= 3\left(e - 1 - \frac{1}{\lfloor 1} - \frac{1}{\lfloor 2}\right) - 3\left(e - 1 - \frac{1}{\lfloor 1}\right) + 2(e-1)$$

$$= \cancel{3e} - \cancel{3} - \cancel{\frac{3}{\lfloor 1}} - \frac{3}{\lfloor 2} + \cancel{3} - \cancel{3e} + \cancel{\frac{3}{\lfloor 1}} + 2e - 2$$

$$= -\frac{3}{\lfloor 2} + 2e - 2$$

$$= 2e - \frac{3}{2} - 2$$

$$S_\infty = 2e - \frac{7}{2}$$

Example 17

Sum to infinity the series

$$\frac{1^2}{\lfloor 3} + \frac{2^2}{\lfloor 5} + \frac{3^2}{\lfloor 7} + \dots \infty$$

$$T_n = \frac{n^2}{\lfloor 2n+1}$$

Let $n^2 = A + B(2n+1) + C(2n+1).2n$

Put $n = -1/2$;

$$\boxed{A = \frac{1}{4}}$$

Put $n = 0$

$$A + B = 0$$

$$\boxed{B = -\frac{1}{4}}$$

Equating the coefficient of n^2, $\boxed{c = 1/4}$

$$\therefore T_n = \frac{1}{4}\cdot\frac{1}{\lfloor\underline{2n+1}} - \frac{1}{4}\frac{1}{\lfloor\underline{2n}} + \frac{1}{4}\cdot\frac{1}{\lfloor\underline{2n-1}}$$

$$\therefore S_\infty = \sum_{n=1}^{\infty} T_n$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{\lfloor\underline{2n+1}} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{\lfloor\underline{2n}} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{\lfloor\underline{2n-1}}$$

$$= \frac{1}{4}\left(\frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{5}} + \frac{1}{\lfloor\underline{7}} + \cdots\right)$$

$$- \frac{1}{4}\left(\frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{4}} + \frac{1}{\lfloor\underline{6}} + \cdots\right) + \frac{1}{4}\left(\frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{5}} + \cdots\right)$$

$$= \frac{1}{4}\left[\frac{e-e^{-1}}{2} - 1\right] - \frac{1}{4}\left[\frac{e+e^{-1}}{2} - 1\right] + \frac{1}{4}\left[\frac{e-e^{-1}}{2}\right]$$

$$= \frac{1}{4}\left(\frac{e-e^{-1}-2}{2}\right) - \frac{1}{4}\left[\frac{e+e^{-1}-2}{2}\right] + \frac{1}{4}\left[\frac{e-e^{-1}}{2}\right]$$

$$= \frac{1}{4}\left(\frac{e-e^{-1}}{2}\right) - \cancel{\frac{1}{4}} - \frac{1}{4}\left(\frac{e+e^{-1}}{2}\right) + \cancel{\frac{1}{4}} + \frac{1}{4}\left(\frac{e-e^{-1}}{2}\right)$$

$$= \frac{\cancel{2}}{4}\left(\frac{e-e^{-1}}{\cancel{2}}\right) - \frac{1}{8}\left(e + \frac{1}{e}\right)$$

$$= \frac{e}{4} - \frac{1}{4e} - \frac{e}{8} - \frac{1}{8e}$$

$$= \frac{e}{8} - \frac{3}{8e}$$

$$= \frac{1}{8}\left(e - \frac{3}{e}\right)$$

$$S_\infty = \frac{1}{8}\left(\frac{e^2-3}{e}\right)$$

Example 18

Find the coefficient of x^r in the expansion of $\dfrac{e^{nx}-1}{1-e^{-x}}$, n being a positive integer and hence find the value of

(i) $1^2+2^2+3^2+\dots+n^2$.

(ii) $1^3+2^3+\dots+n^3$.

$$\frac{e^{nx}-1}{1-e^{-x}} = \frac{e^{nx}-1}{1-\frac{1}{e^x}}$$

$$= \frac{e^x(e^{nx}-1)}{e^x-1}$$

$$= e^x\left[e^{(n-1)x}+e^{(n-2)x}+e^{(n-3)x}+\dots + e^{[n-(n-1)]x}+e^{(n-n)x}\right]$$

$$= e^x\left[e^{(n-1)x}+e^{(n-2)x}+e^{(n-3)x}+\dots+e^x+1\right]$$

$$= e^x . e^{nx-x}+e^x . e^{nx-2x}+e^x . e^{nx-3x}+\dots + e^x . e^x+e^x$$

$$= e^{nx}+e^{(n-1)x}+e^{(n-2)x}+\dots+e^{2x}+e^x$$

Coefficient of x^r in the above expansion n

$$\frac{e^{nx}-1}{1-e^{-x}} = \frac{1}{r!}\left[n^r+(n-1)^r+(n-2)^r+\dots+2^r+1^r\right] \rightarrow (1)$$

Also $$\frac{e^{nx}-1}{1-e^{-x}} = \frac{nx+\frac{n^2x^2}{2!}+\frac{n^3x^3}{3!}+\dots\infty}{x-\frac{x^2}{2!}+\frac{x^3}{3!}-\dots\infty}$$

$e^{-x} = \dots$ (and) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\dots\infty$ formulae used.

$$\cancel{e^{nx}-1}$$

$$\frac{e^{nx}-1}{1-e^{-x}} = \frac{n\cancel{x}\left[1+\frac{nx}{2}+\frac{n^2x^2}{6}+\frac{n^3x^3}{24}+\dots\infty\right]}{\cancel{x}\left[1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^3}{24}+\dots\infty\right]}$$

$$= n\left[1+\frac{nx}{2}+\frac{n^2x^2}{6}+\dots\infty\right]\left[1-\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}\dots\infty\right)\right]^{-1}$$

$$= n\left[1+\frac{nx}{2}+\frac{n^2x^2}{6}+\frac{n^3x^3}{24}+\dots\infty\right]\left[1+\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\dots\right)\right.$$

$$\left.+\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\dots\right)^2+\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\dots\right)^3+\dots\infty\right]$$

$(1-x)^{-1} = 1+x+x^2+x^3+\dots\infty$ formula used.

$$\frac{e^{nx}-1}{1-e^{-x}} = n\left[1+\frac{nx}{2}+\frac{n^2x^2}{6}+\frac{n^3x^3}{24}+\dots\infty\right]$$

$$\left[1+\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}+\frac{x^2}{4}-\frac{2x^3}{12}+\frac{x^3}{8}\right]$$

x^4 and higher terms are neglated.

$$\frac{e^{nx}-1}{1-e^{-x}} = n\left[1+\frac{nx}{2}+\frac{n^2x^2}{6}+\dots\infty\right]\left[1+\frac{x}{2}+\frac{3x^2-2x^2}{12}\right.$$

$$\left.+\frac{\cancel{x^3}-\cancel{4x^3}+\cancel{3x^3}}{24}\right]$$

$$= n\left[1+\frac{nx}{2}+\frac{n^2x^2}{6}+\dots\infty\right]\left[1+\frac{x}{2}+\frac{x^2}{12}+0\right]$$

$$= n\left[1+\frac{x}{2}+\frac{x^2}{12}+\frac{nx}{2}+\frac{nx^2}{4}+\frac{nx^3}{24}+\frac{n^2x^2}{6}+\frac{n^2x^3}{12}+\dots\right.$$

$$\left.+\frac{n^3x^3}{24}\right]$$

$$\frac{e^{nx}-1}{1-e^{-x}} = n\left[1+\frac{x}{2}+\frac{x^2}{12}+\frac{nx}{2}+\frac{nx^2}{4}+\frac{nx^3}{24}+\frac{n^2x^2}{6}+\frac{n^2x^3}{12}+\frac{n^3x^3}{24}\right]$$

$$= n\left[1+\frac{x+nx}{2}+\frac{x^2+3nx^2+2n^2x^2}{12}+\frac{nx^3+2n^2x^3+n^3x^3}{24}+\dots\right]$$

$$= n\left[1+\frac{(n+1)x}{2}+\frac{(2n^2+3n+1)}{12}x^2+\frac{(n^3+2n^2+n)}{24}x^3+\dots\right] \to ②$$

Equating the coefficient of x^2 and x^3 in ① and ②

$$\frac{1}{\cancel{\lfloor 2}}\left[1^2+2^2+\dots+n^2\right] = n\frac{(2n^2+3n+1)}{\cancel{12}\,6} \to ③$$

$$\frac{1}{\cancel{\lfloor 3}}\left[1^3+2^3+\dots+n^3\right] = n\frac{(n^3+2n^2+n)}{\cancel{24}\,4} \to ④$$

From ③, $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

From ④, $1^3+2^3+3^3+\dots n^3 = \frac{n^2(n^2+2n+1)}{4}$

$$1^3+2^3+3^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$$

(or) $\left[\frac{n(n+1)}{2}\right]^2$